Lagrange's equation of motion from D' Alembert's Principle:

The co-ordinate transformation equations are:

$$\vec{r_i} = \vec{r_i}(q_1, q_2, \ldots, q_n, t)$$

So that,

$$\frac{d\vec{r_i}}{dt} = \frac{\partial \vec{r_i}}{\partial q_1}\cdot\frac{dq_1}{dt} + \frac{\partial \vec{r_i}}{\partial q_2}\cdot\frac{dq_2}{dt} + \ldots + \frac{\partial \vec{r_i}}{\partial t}\cdot\frac{dt}{dt}$$

or,

$$\vec{v_i} = \sum_j \frac{\partial \vec{r_i}}{\partial q_j}\cdot\dot{q_j} + \frac{\partial \vec{r_i}}{\partial t} \qquad (3)$$

Further infinitesimal displacement $\delta\vec{r_i}$ can be connected with δq_j as

$$\delta \vec{r}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \cdot \delta q_j + \frac{\partial \vec{r}_i}{\partial t} \delta t$$

But last term is zero since in virtual displacement only co-ordinate displacement is considered and not that of time. Therefore,

$$\delta \vec{r}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \cdot \delta q_j$$

Now, we write eq^n (2) as,

$$\sum_i (\vec{F}_i - \vec{\dot{p}}_i) \cdot \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \cdot \delta q_j = 0,$$

$$\sum_{ij} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \cdot \delta q_j - \sum_{ij} \vec{\dot{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \cdot \delta q_j = 0.$$

We define the term

$$\sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = Q_j$$

the components of the generalised force.

The product $Q_j \delta q_j$ must have the dimensions of work. Thus, above equation takes the form

$$\sum_j Q_j \cdot \delta q_j - \sum_{ij} \vec{\dot{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j = 0 \quad (4)$$

Let us evaluate the second term of above eq. (4):

$$\sum_{ij} \vec{p} \cdot \frac{\partial \vec{r_i}}{\partial q_j} \cdot \delta q_j = \sum_{ij} m_i \dot{\vec{r_i}} \cdot \frac{\partial \vec{r_i}}{\partial q_j} \cdot \delta q_i$$

$$= \sum_{ij} \left\{ \frac{d}{dt}\left(m_i \dot{\vec{r_i}} \cdot \frac{\partial \vec{r_i}}{\partial q_j} \right) - m_i \dot{\vec{r_i}} \cdot \frac{d}{dt}\left(\frac{\partial \vec{r_i}}{\partial q_j} \right) \right\} \delta q_j$$

$$= \sum_{ij} \left\{ \frac{d}{dt}\left(m_i \vec{v_i} \cdot \frac{\partial \vec{r_i}}{\partial q_j} \right) - m_i \vec{v_i} \cdot \frac{d}{dt}\left(\frac{\partial \vec{r_i}}{\partial q_j} \right) \right\} \delta q_j \quad \text{--- (5)}$$

further,

$$\frac{d}{dt}\left(\frac{\partial \vec{r_i}}{\partial q_j} \right) = \sum_k \frac{\partial}{\partial q_k}\left(\frac{\partial \vec{r_i}}{\partial q_j} \right) \cdot \frac{dq_k}{dt} + \frac{\partial}{\partial t}\left(\frac{\partial \vec{r_i}}{\partial q_j} \right) \cdot \frac{dt}{dt}$$

$$= \sum_k \frac{\partial^2 \vec{r_i}}{\partial q_k \cdot \partial q_j} \cdot \dot{q}_k + \frac{\partial^2 \vec{r_i}}{\partial t \cdot \partial q_j}$$

$$= \sum_k \frac{\partial^2 \vec{r_j}}{\partial q_j \cdot \partial q_k} \cdot \dot{q}_k + \frac{\partial^2 \vec{r_i}}{\partial q_j \cdot \partial t}$$

$$= \frac{\partial}{\partial q_j}\left(\frac{d \vec{r_i}}{dt} \right) = \frac{\partial \vec{v_i}}{\partial q_j} \quad \text{--- (6)}$$

Also differentiating eq^n (3) with respect to $\dot{q}_j$,

we get
$$\frac{\partial \vec{v}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j} \quad \text{--- (7)}$$

Putting eq^n (6) and (7) in eq^n (5),
we get

$$\sum_{ij} \vec{p}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \cdot \delta q_j = \sum_{ij} \left[\frac{d}{dt} \left(m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j} \right) - m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_j} \right] \delta q_j$$

$$= \sum_j \left[\frac{d}{dt} \left\{ \frac{\partial}{\partial \dot{q}_j} \left(\sum_i \frac{1}{2} m_i v_i^2 \right) \right\} - \frac{\partial}{\partial q_j} \left(\sum_i \frac{1}{2} m_i v_i^2 \right) \right] \delta q_j$$

With this substitution, eq^n (4) becomes

$$\sum_j Q_j \delta q_j - \sum_j \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] \delta q_j = 0$$

or,
$$\sum_j \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j \right] \delta q_j = 0$$

where, the term $\sum_i \frac{1}{2} m_i v_i^2$, represents the total Kinetic energy T of the System.

Since the constraints are holonomic q_j are independent of each other and hence to satisfy above equation the coefficient of each δq_j should separately Vanish, i.e.

$$\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right] = Q_j \quad \text{--- (8)}$$

As j ranges from 1 to N, there will be N such second order equations.

<u>Case:-</u> Lagrange's equation of motion for <u>Conservative system</u>:

We Consider a Conservative system and use above set of equations (8) to obtain Lagrange's equations of motion. For a Conservative system, forces $\vec{F_i}$ are derivable from potential function V, which is the function of Co-ordinates only

That is $V = V(\vec{r})$, then

$$\vec{F_i} = -\nabla_i V$$

$$= -\frac{\partial V}{\partial \vec{r_i}}$$

Then generalised force can be expressed as:

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

$$= -\sum_i \nabla_i V \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

$$= -\sum_i \frac{\partial V}{\partial \vec{r}_i} \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

$$= -\frac{\partial V}{\partial q_j}$$

Eqs. (8) are now

$$\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right] = -\frac{\partial V}{\partial q_j}$$

or, $$\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}(T-V)\right] = 0$$

or, $$\left[\frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j}\right] = 0$$

Since V is not a function of $\dot{q}_j$.

Recognising $(T-V)$ as L, the Lagrangian for the conservative system, equation become,

$$\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}\right] = 0,$$

$$j = 1, 2 \cdots n \qquad (9)$$

which are known as Lagrange's equations of motion for conservative system.